

Minimum-error discrimination between mixed quantum states

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Abstract

We derive a general lower bound on the minimum-error probability for *ambiguous discrimination* between arbitrary m mixed quantum states with given prior probabilities. When $m = 2$, this bound is precisely the well-known Helstrom limit. Also, we give a general lower bound on the minimum-error probability for discriminating quantum operations. Then we further analyze how this lower bound is attainable for ambiguous discrimination of mixed quantum states by presenting necessary and sufficient conditions related to it. Furthermore, with a restricted condition, we work out a upper bound on the minimum-error probability for ambiguous discrimination of mixed quantum states. Therefore, some sufficient conditions are obtained for the minimum-error probability attaining this bound. Finally, under the condition of the minimum-error probability attaining this bound, we compare the minimum-error probability for *ambiguously* discriminating arbitrary m mixed quantum states with the optimal failure probability for *unambiguously* discriminating the same states.

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I. Introduction

Motivated by the study of quantum communication and quantum cryptography, distinguishing quantum states has become a fundamental subject in quantum information science [1,2]. This problem may be roughly described in this manner [1,2,3,4,5,6]: Suppose that a transmitter, Alice, wants to convey classical information to a receiver, Bob, using a quantum channel, and Alice represents the message conveyed as a mixed quantum state that, with given prior probabilities, belongs to a finite set of mixed quantum states, say $\{\rho_1, \rho_2, \dots, \rho_m\}$; then Bob identifies the state by a measurement.

As it is known [4,5,6], if the supports of mixed states $\rho_1, \rho_2, \dots, \rho_m$ are not mutually orthogonal, then Bob can not reliably identify which state Alice has sent, namely, $\rho_1, \rho_2, \dots, \rho_m$ can not be faithfully distinguished. However, it is always possible to discriminate them in a probabilistic means. In reality, up to now, various strategies have been proposed for distinguishing quantum states. Assume that mixed states $\rho_1, \rho_2, \dots, \rho_m$ have the *a priori* probabilities $\eta_1, \eta_2, \dots, \eta_m$, respectively. In general, there are three fashions to discriminate them. The first approach is *ambiguous discrimination* [4,5,6], in which inconclusive outcome is not allowed, and thus error may result. A measurement for discrimination consists of m measurement operators (e.g., positive semidefinite operators) that form a resolution of the identity on the Hilbert space spanned by the all eigenvectors corresponding to all nonzero eigenvalues of $\rho_1, \rho_2, \dots, \rho_m$. Much work has been devoted to devising a measurement maximizing the success probability for detecting the states [7,8,9,10,11]. The first important result is the pioneering work by Helstrom [4]—a general expression of the minimum achievable error probability for distinguishing between two mixed quantum states. For the case of more than two quantum states, necessary and sufficient conditions have been derived for an optimum measurement maximizing the success probability of correct detection [5,6,8]. However, analytical solutions for an optimum measurement have been obtained only for some special cases [11,12,13,14,15], and, as pointed out in [8], obtaining a concrete expression for an optimum measurement in the general case is a difficult and unsolved problem.

The second approach is the so-called *unambiguous discrimination* [1,2,16-26], first suggested by Ivanovic, Dicks, and Peres [16,17,18] for the discrimination of two pure states. In

contrast to ambiguous discrimination, unambiguous discrimination allows an inconclusive result to be returned, but no error occurs. In other words, for distinguishing between m mixed states, this basic idea is to devise a measurement that with a probability returns an inconclusive result, but, if the measurement returns an answer, then the answer is fully correct. Therefore, such a measurement consists of $m + 1$ measurement operators, in which a measurement operator returns an inconclusive outcome. Analytical solutions for the optimal failure probabilities have been given for distinguishing between two and three pure states [16,17,18,19,20,21]. Chefles [22] showed that a set $\{|\psi\rangle\}$ of pure states is amenable to unambiguous discrimination if, and only if they are linearly independent. The optimal unambiguous discrimination between linearly independent symmetric and equiprobable pure states was solved in [23]. By means of Lagrange multiplier, Sun *et al.* [24] presented a scheme for calculating the optimal probability of unambiguous discrimination among linearly independent, nonorthogonal pure states. A semidefinite programming approach to unambiguous discrimination between pure states has been investigated in detail by Eldar [25]. Some upper bounds on the success probability for unambiguous discrimination between pure states have also been presented (see [26] and references therein).

We recollect unambiguous discrimination between mixed quantum states. For distinguishing between two mixed quantum states, general upper and lower bounds have been derived for the optimal failure probability by Rudolph *et al.* [27], and, furthermore, for distinguishing between m mixed states, Feng *et al.* [28] obtained a general lower bound on the minimum failure probability. The analytical results for the optimal unambiguous discrimination between two mixed quantum states have been derived by Raynal *et al.* [29], by Herzog and Bergou [30], and by Zhou *et al.* [31]. More references regarding unambiguous discrimination of mixed quantum states may be referred to [32]. (It is also worth mentioning that a universal programmable quantum device has been designed recently for unambiguous discrimination of pure states [33], and such a device can be considered for discriminating mixed states.)

The third strategy for discrimination combines the former two methods [34,35,36]. That is to say, under the condition that a fixed probability of inconclusive outcome is allowed to occur, one tries to determine the minimum achievable probability of errors for ambiguous

discrimination. Chefles *et al.* [34] and Fiurášek *et al.* [35] considered the case of discriminating pure states, and Eldar [36] dealt with this discrimination of mixed states. Indeed, by allowing for an inconclusive result occurring, then one can obtain a higher probability of correct detection for getting a conclusive result, than the probability of correct detection attainable without inconclusive results appearing.

In general, the above discrimination schemes are assumed to have *a priori* probabilities for the states to be discriminated. Notably, a different scheme recently addressed by D'Ariano *et al.* [37] is the minimax quantum state discrimination strategy, in which the optimal measurement has been derived for mixed state discrimination without *a priori* probabilities.

In this paper, we deal with ambiguous discrimination between any m mixed quantum states and compare with unambiguous discrimination. The main contributions include three aspects: First we derive a general lower bound on the minimum-error probability for distinguishing between any m mixed quantum states. When $m = 2$, this lower bound is precisely the well-known Helstrom limit [4]. Therefore, in the case of discriminating two mixed states, this bound can always be achieved. By means of the lower bound, we further give a lower bound on the minimum-error probability for discriminating quantum operations. Then we further analyze how this lower bound is attainable for ambiguous discrimination of mixed states by presenting necessary and sufficient conditions related to it. Furthermore, with a restricted condition, we work out an upper bound on the minimum-error probability for ambiguous discrimination of mixed states. Therefore, some sufficient conditions are obtained for the minimum-error probability attaining this bound. Finally, under the condition of the minimum-error probability attaining this bound, we compare the minimum-error probability for *ambiguously* discriminating arbitrary m mixed states with the optimal failure probability for *unambiguously* discriminating the same mixed states. When $m = 2$, this result has been proved by Herzeg and Bergou [38].

The remainder of the paper is organized as follows. In Section II, we derive a lower bound on the minimum-error probability for ambiguous discrimination between arbitrary m mixed states. With this bound, we give a lower bound on the minimum-error probability for discriminating quantum operations. Then, in Section III, we further analyze the reachability

of this lower bound derived in Section II, and, in Subsection A, we show some necessary and sufficient conditions related to it. Furthermore, in Subsection B, with a restricted condition, we work out a upper bound on the minimum-error probability. After that, in Section IV, we deal with the relation between the minimum-error probability for ambiguous discrimination of mixed states and the optimal failure probability for unambiguous discrimination of the same mixed states. Finally, some concluding remarks are made in Section V.

In general, notation used in this paper will be explained whenever new symbols appear. Here we first give a denotation that will be useful in what follows: For any two linear operators T_1 and T_2 on the same Hilbert space \mathcal{H} , we use $T_1 \perp T_2$ to denote that the supports of T_1 and T_2 are orthogonal. The support of a linear operator T is the subspace spanned by the all eigenvectors corresponding to all nonzero eigenvalues of T .

II. A lower bound on the minimum-error discrimination between mixed quantum states

Assume that a quantum system is described by a mixed quantum state, say ρ , drawn from a collection $\{\rho_1, \rho_2, \dots, \rho_m\}$ of mixed quantum states on an n -dimensional complex Hilbert space \mathcal{H} , with the *a priori* probabilities $\eta_1, \eta_2, \dots, \eta_m$, respectively, where $m \leq n$. We assume without loss of generality that the all eigenvectors of ρ_i , $1 \leq i \leq m$, span \mathcal{H} , otherwise we consider the spanned subspace instead of \mathcal{H} . A mixed quantum state ρ is a positive semidefinite operator with trace 1, denoted $\text{Tr}(\rho) = 1$. (Note that a positive semidefinite operator must be a Hermitian operator [39,40].) To detect ρ , we need to design a measurement consisting of m positive semidefinite operators, say Π_i , $1 \leq i \leq m$, satisfying the resolution

$$\sum_{i=1}^m \Pi_i = I, \quad (1)$$

where I denotes the identity operator on \mathcal{H} . By means of the measurement Π_i , $1 \leq i \leq m$, if the system has been prepared by ρ , then $\text{Tr}(\rho\Pi_i)$ is the probability to deduce the system being state ρ_i . Therefore, the average probability P of correct detecting the system's state

is as follows:

$$P = \sum_{i=1}^m \eta_i \text{Tr}(\rho_i \Pi_i) \quad (2)$$

and, the average probability Q of erroneous detection is then as

$$Q = 1 - P = 1 - \sum_{i=1}^m \eta_i \text{Tr}(\rho_i \Pi_i). \quad (3)$$

A main objective is to design an optimum measurement that minimizes the probability of erroneous detection. As mentioned above, for the case of $m = 2$, the optimum detection problem has been completely solved by Helstrom [4], and the minimum achievable error probability, say Q_A , has been presented by the Helstrom limit [4]

$$Q_A = \frac{1}{2}(1 - \text{Tr}|\eta_2 \rho_2 - \eta_1 \rho_1|), \quad (4)$$

where $|A| = \sqrt{A^\dagger A}$ for any linear operator A , and A^\dagger denotes the conjugate transpose of A .

However, for $m > 2$, the problem is much more complicated, and, as indicated above, there has not been a general analytical expression for the minimum-error probability for ambiguously distinguishing between arbitrary m mixed states. To this end, we show a general analytical solution to a lower bound on the minimum-error probability for ambiguously distinguishing between arbitrary m mixed quantum states. Then we will analyze this bound. We first present a lemma that is useful in the paper.

Lemma 1. Let A and B be two positive semidefinite operators. Then $\text{Tr}|A - B| \leq \text{Tr}(A) + \text{Tr}(B)$, and the equality holds if, and only if $A \perp B$.

Proof. Suppose that A and B have the following spectral decompositions:

$$A = \sum_{i=1}^{k_1} \lambda_i |\lambda_i\rangle \langle \lambda_i|, \quad (5)$$

$$B = \sum_{j=1}^{k_2} \mu_j |\mu_j\rangle \langle \mu_j|, \quad (6)$$

where all $\lambda_i > 0$ and all $\mu_j > 0$.

If $A \perp B$, then $\langle \lambda_i | \mu_j \rangle = 0$ for $1 \leq i \leq k_1$ and $1 \leq j \leq k_2$, and thus $AB = BA = \mathbf{0}$, where $\mathbf{0}$ denotes a zero operator. In this case, we obtain

$$|A - B| = \sqrt{(A - B)^\dagger (A - B)} \quad (7)$$

$$= \sqrt{A^2 + B^2 - (AB + BA)} \quad (8)$$

$$= \sqrt{A^2 + B^2} \quad (9)$$

$$= A + B, \quad (10)$$

where Eq. (10) is due to $A \perp B$. Consequently, $\text{Tr}|A - B| = \text{Tr}(A) + \text{Tr}(B)$.

Before the following proof, we recall some properties of trace distance and fidelity. Indeed, as we know from [40], for any mixed states ρ and σ , the trace distance $D(\rho, \sigma)$ and fidelity $F(\rho, \sigma)$ satisfy:

$$D(\rho, \sigma) = \max_{\{E_m\}} \frac{1}{2} \sum_m |\text{Tr}(E_m \rho) - \text{Tr}(E_m \sigma)|, \quad (11)$$

and

$$F(\rho, \sigma) = \min_{\{E_m\}} \sum_m \sqrt{\text{Tr}(E_m \rho) \text{Tr}(E_m \sigma)}, \quad (12)$$

where the maximum and the minimum are over all POVMs $\{E_m\}$, $F(\rho, \sigma) = \text{Tr} \sqrt{(\rho^{1/2} \sigma \rho^{1/2})}$ and $D(\rho, \sigma) = \frac{1}{2} \text{Tr} |\rho - \sigma|$. In fact, in the proof for Eqs. (11,12) in [40], the traces of ρ and σ being one is not involved, and it only utilizes the positive semidefinite property of ρ and σ . Therefore, for any positive semidefinite operators A and B , Eqs. (11,12) hold as well, whose proof is only a repeated process step by step according to those of [40]. In other words, for any positive semidefinite operators A and B , we also have

$$D(A, B) = \max_{\{E_m\}} \frac{1}{2} \sum_m |\text{Tr}(E_m A) - \text{Tr}(E_m B)|, \quad (13)$$

and

$$F(A, B) = \min_{\{E_m\}} \sum_m \sqrt{\text{Tr}(E_m A) \text{Tr}(E_m B)}, \quad (14)$$

where the maximum and the minimum are over all POVMs $\{E_m\}$, $F(A, B) = \text{Tr} \sqrt{(A^{1/2} B A^{1/2})}$ and $D(A, B) = \frac{1}{2} \text{Tr} |A - B|$.

Therefore, we always have

$$\text{Tr}|A - B| = \max_{\{E_m\}} \sum_m |\text{Tr}(E_m A) - \text{Tr}(E_m B)| \quad (15)$$

$$\leq \max_{\{E_m\}} \left(\sum_m \text{Tr}(E_m A) + \sum_m \text{Tr}(E_m B) \right) \quad (16)$$

$$= \text{Tr}(A) + \text{Tr}(B), \quad (17)$$

where Eq. (17) is due to $\sum_m E_m = I$ for any POVM $\{E_m\}$.

If $\text{Tr}|A - B| = \text{Tr}(A) + \text{Tr}(B)$ holds, we claim $A \perp B$. Indeed, by means of Eq. (13) there is a POVM, say $\{\Pi_m\}$ such that

$$D(A, B) = \frac{1}{2} \sum_m |\text{Tr}(\Pi_m A) - \text{Tr}(\Pi_m B)|, \quad (18)$$

from which we have

$$\text{Tr}|A - B| = \sum_m |\text{Tr}(\Pi_m A) - \text{Tr}(\Pi_m B)| \quad (19)$$

$$\leq \sum_m \text{Tr}(\Pi_m A) + \sum_m \text{Tr}(\Pi_m B) \quad (20)$$

$$= \text{Tr}(A) + \text{Tr}(B). \quad (21)$$

Since we assume $\text{Tr}|A - B| = \text{Tr}(A) + \text{Tr}(B)$, inequality (20) must be an equality, which implies that, for each m , $\text{Tr}(\Pi_m A) = 0$ or $\text{Tr}(\Pi_m B) = 0$. Thus, for each m , we have $\text{Tr}(\Pi_m A)\text{Tr}(\Pi_m B) = 0$. As a result,

$$F(A, B) = \min_{\{E_m\}} \sum_m \sqrt{\text{Tr}(E_m A)\text{Tr}(E_m B)} \quad (22)$$

$$\leq \sum_m \sqrt{\text{Tr}(\Pi_m A)\text{Tr}(\Pi_m B)} \quad (23)$$

$$= 0. \quad (24)$$

Consequently, $F(A, B) = 0$, i.e., $\text{Tr}\sqrt{(A^{1/2}BA^{1/2})} = 0$. Therefore, due to $A^{1/2}BA^{1/2}$ being a positive semidefinite operator, $A^{1/2}BA^{1/2}$ is a zero operator. Then $A \perp B$ must hold. Otherwise, there is at least a pair (i_0, j_0) such that

$$\langle \lambda_{i_0} | \mu_{j_0} \rangle \neq 0. \quad (25)$$

Further, by means of Eqs. (5,6), we have

$$\langle \lambda_{i_0} | A^{1/2}BA^{1/2} | \lambda_{i_0} \rangle = \lambda_{i_0} \langle \lambda_{i_0} | B | \lambda_{i_0} \rangle \quad (26)$$

$$\geq \lambda_{i_0} \mu_{j_0} |\langle \mu_{j_0} | \lambda_{i_0} \rangle|^2 \quad (27)$$

$$> 0, \quad (28)$$

which contradicts $A^{1/2}BA^{1/2}$ being a zero operator. Therefore, we have shown that $\text{Tr}|A - B| = \text{Tr}(A) + \text{Tr}(B)$ implies $A \perp B$. This has completed the proof. \square

Now we present the following theorem.

Theorem 1. For any m mixed quantum states $\rho_1, \rho_2, \dots, \rho_m$, with the *a priori* probabilities $\eta_1, \eta_2, \dots, \eta_m$, respectively, then the minimum-error probability Q_A satisfies

$$Q_A \geq \frac{1}{2} \left(1 - \frac{1}{m-1} \sum_{1 \leq i < j \leq m} \text{Tr} |\eta_j \rho_j - \eta_i \rho_i| \right). \quad (29)$$

Proof. It suffices to show that the maximum probability, say P_A , of correct detection satisfies

$$P_A \leq \frac{1}{2} \left(1 + \frac{1}{m-1} \sum_{1 \leq i < j \leq m} \text{Tr} |\eta_j \rho_j - \eta_i \rho_i| \right). \quad (30)$$

For convenience, we first give two symbols: $\mathcal{M} = \{ \{ \Pi_i \}_{i=1}^m : \sum_{i=1}^m \Pi_i = I \}$ where Π_i are positive semidefinite operators; and we denote $\Lambda_{ij} = \eta_j \rho_j - \eta_i \rho_i$ in this paper.

According to Eqs. (1,2), we know

$$P_A = \max_{\{ \Pi_i \}_{i=1}^m} \sum_{i=1}^m \text{Tr} (\eta_i \rho_i \Pi_i), \quad (31)$$

where the maximization is performed over all POVMs $\{ \Pi_i \}_{i=1}^m \in \mathcal{M}$. By the way, from the theoretical point of view [6,7,8,9,10], the “max” does exist in Eq. (31), so, we can use “max” instead of “sup”. Of course, this representation is independent of our proof and result.

Note that

$$(m-1) \sum_{i=1}^m \text{Tr} (\eta_i \rho_i \Pi_i) + \sum_{\substack{1 \leq i < j \leq m \\ k \neq i, j}} (\eta_i \text{Tr} (\rho_i \Pi_k)) = \sum_{1 \leq i < j \leq m} [\eta_i + \text{Tr} (\Lambda_{ij} \Pi_j)], \quad (32)$$

where $\sum_{i=1}^m \eta_i = 1$ is used. We know that any Hermitian operator H can be represented as the form $H = A - B$ where A and B are positive semidefinite operators and $A \perp B$ (i.e., the supports of A and B are orthogonal). Indeed, the spectral decomposition of H readily verifies this fact. Since Λ_{ij} is Hermitian, we let

$$\Lambda_{ij} = A_{ij} - B_{ij} \quad (33)$$

where A_{ij} and B_{ij} are positive semidefinite operators with $A_{ij} \perp B_{ij}$. In addition, we represent them with their spectral decomposition forms

$$A_{ij} = \sum_k a_k^{(ij)} |\phi_k^{(ij)}\rangle \langle \phi_k^{(ij)}|, \quad (34)$$

$$B_{ij} = \sum_l b_l^{(ij)} |\varphi_l^{(ij)}\rangle \langle \varphi_l^{(ij)}|, \quad (35)$$

where $|\phi_k^{(ij)}\rangle$ and $|\varphi_l^{(ij)}\rangle$ are mutually orthogonal for all k and l , and $a_k^{(ij)}$, $b_l^{(ij)}$ are positive real numbers. With Eqs. (32,33,34,35) we have

$$\begin{aligned} & \sum_{i=1}^m \text{Tr}(\eta_i \rho_i \Pi_i) \\ &= \frac{1}{m-1} \sum_{1 \leq i < j \leq m} [\eta_i + \text{Tr}(A_{ij} \Pi_j) - \text{Tr}(B_{ij} \Pi_j)] - \frac{1}{m-1} \sum_{\substack{1 \leq i < j \leq m \\ k \neq i, j}} (\eta_i \text{Tr}(\rho_i \Pi_k)) \end{aligned} \quad (36)$$

$$\leq \frac{1}{m-1} \sum_{1 \leq i < j \leq m} [\eta_i + \sum_k a_k^{(ij)} \langle \phi_k^{(ij)} | \Pi_j | \phi_k^{(ij)} \rangle - \sum_l b_l^{(ij)} \langle \varphi_l^{(ij)} | \Pi_j | \varphi_l^{(ij)} \rangle] \quad (37)$$

$$\leq \frac{1}{m-1} \sum_{1 \leq i < j \leq m} [\eta_i + \sum_k a_k^{(ij)}], \quad (38)$$

where Ineq. (37) is due to $\sum_{\substack{1 \leq i < j \leq m \\ k \neq i, j}} (\eta_i \text{Tr}(\rho_i \Pi_k)) \geq 0$. Next we show that

$$\frac{1}{2} \left(1 + \frac{1}{m-1} \sum_{1 \leq i < j \leq m} \text{Tr}|\Lambda_{ij}| \right) = \frac{1}{m-1} \sum_{1 \leq i < j \leq m} (\eta_i + \sum_k a_k^{(ij)}). \quad (39)$$

By combining $\text{Tr}(\Lambda_{ij}) = \eta_j - \eta_i$ with Eqs. (33,34,35), we have

$$\text{Tr}(\Lambda_{ij}) = \eta_j - \eta_i = \sum_k a_k^{(ij)} - \sum_l b_l^{(ij)}. \quad (40)$$

Since $A_{ij} \perp B_{ij}$, with Lemma 1 and Eqs. (33,34,35) we further have

$$\text{Tr}|\Lambda_{ij}| = \text{Tr}(A_{ij}) + \text{Tr}(B_{ij}) = \sum_k a_k^{(ij)} + \sum_l b_l^{(ij)}. \quad (41)$$

Therefore, with Eqs. (40,41) we obtain

$$\begin{aligned} & \frac{1}{2} \left(1 + \frac{1}{m-1} \sum_{1 \leq i < j \leq m} \text{Tr}|\Lambda_{ij}| \right) \\ &= \frac{1}{2} \left[1 + \frac{1}{m-1} \sum_{1 \leq i < j \leq m} \left(\sum_k a_k^{(ij)} + \sum_l b_l^{(ij)} \right) \right] \end{aligned} \quad (42)$$

$$= \frac{1}{2} \left[1 + \frac{1}{m-1} \sum_{1 \leq i < j \leq m} (2 \sum_k a_k^{(ij)} + \eta_i - \eta_j) \right] \quad (43)$$

$$= \frac{1}{2} + \frac{1}{m-1} \sum_{1 \leq i < j \leq m} (\eta_i + \sum_k a_k^{(ij)}) - \frac{1}{2(m-1)} \sum_{1 \leq i < j \leq m} (\eta_i + \eta_j) \quad (44)$$

$$= \frac{1}{m-1} \sum_{1 \leq i < j \leq m} (\eta_i + \sum_k a_k^{(ij)}), \quad (45)$$

where the last equality results from

$$\frac{1}{m-1} \sum_{1 \leq i < j \leq m} (\eta_i + \eta_j) = 1. \quad (46)$$

As a result, Eq. (39) holds, and, in terms of Ineq. (38), the theorem has been proved. \square

Remark 1. With Lemma 1, $\text{Tr}|\eta_j \rho_j - \eta_i \rho_i| \leq \eta_j + \eta_i$ and, the equality holds if and only if $\rho_j \perp \rho_i$. In Theorem 1, the upper bound on the probability of correct detection between m mixed quantum states satisfies

$$\begin{aligned} & \frac{1}{2} \left(1 + \frac{1}{m-1} \sum_{1 \leq i < j \leq m} \text{Tr}|\eta_j \rho_j - \eta_i \rho_i| \right) \\ & \leq \frac{1}{2} \left[1 + \frac{1}{m-1} \sum_{1 \leq i < j \leq m} (\eta_j + \eta_i) \right] = 1, \end{aligned} \quad (47)$$

and, by means of Lemma 1, we further see that this bound is strictly smaller than 1 usually unless $\rho_1, \rho_2, \dots, \rho_m$ are mutually orthogonal. \square

Remark 2. When $m = 2$, the lower bound in Theorem 1 is $\frac{1}{2}(1 - \text{Tr}|\eta_2 \rho_2 - \eta_1 \rho_1|)$, which accords with the well-known Helstrom limit [4]; and indeed, in this case, this bound can always be attained by choosing the optimum *positive operator-valued measurement* (POVM): $\Pi_2 = \sum_k |\phi_k^{(12)}\rangle \langle \phi_k^{(12)}|$ and $\Pi_1 = I - \Pi_2$. \square

Remark 3. From Theorem 1 it readily follows a lower bound on the minimum-error probability for discriminating m quantum operations. With respect to quantum operations, we refer to [40]. The problem of the minimum-error discrimination between two quantum operations, say \mathcal{E}_1 and \mathcal{E}_2 , with given prior probabilities η_1, η_2 , respectively, has been formulated by Sacchi [41]. The minimum-error probability, say Q_E , equals

$$Q_E = \frac{1}{2} (1 - \max_{\rho} \text{Tr}|\eta_2 \mathcal{E}_2(\rho) - \eta_1 \mathcal{E}_1(\rho)|) \quad (48)$$

where ρ is in the Hilbert space \mathcal{H} under consideration.

Then, for arbitrary m quantum operations $\mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_m$ with the *a priori* probabilities $\eta_1, \eta_2, \dots, \eta_m$, respectively, in terms of a POVM $\{\Pi_i : 1 \leq i \leq m\}$, the probability of erroneous detection is

$$1 - \max_{\rho} \sum_{i=1}^m \eta_i \text{Tr}[\mathcal{E}_i(\rho) \Pi_i], \quad (49)$$

where ρ is in the Hilbert space \mathcal{H} under consideration. Therefore, by means of Theorem 1, the minimum-error probability Q_E for discriminating $\mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_m$ satisfies

$$Q_E \geq \min_{\rho} \frac{1}{2} \left(1 - \frac{1}{m-1} \sum_{1 \leq i < j \leq m} \text{Tr} |\eta_j \mathcal{E}_j(\rho) - \eta_i \mathcal{E}_i(\rho)| \right). \quad (50)$$

□

III. Further analysis on the lower bound

In this section, we analyze how the lower bound derived in Section II can be approached. As we know, for any POVM $\{\Pi_j : 1 \leq j \leq m\}$, $\sum_{\substack{1 \leq i < j \leq m \\ k \neq i, j}} (\eta_i \text{Tr}(\rho_i \Pi_k)) \geq 0$. When $\text{Tr}(\rho_i \Pi_k) = 0$ for all $1 \leq i \leq m-1$ and $k \neq i$, it equals 0, which shows that Ineq. (37) can become an equality in this case (for example, a strong condition is that $\rho_1, \rho_2, \dots, \rho_{m-1}$ are mutually orthogonal).

In Subsection A, we will show in detail that Ineq. (38) in the proof of Theorem 1 can become an equality if and only if $\rho_1, \rho_2, \dots, \rho_m$ satisfy a certain condition. Then, in Subsection B, we deal with Ineq. (37) and determine an upper bound on the minimum-error probability for discriminating m quantum states under certain conditions.

A. Necessary and sufficient conditions concerning inequality (38)

From Ineq. (38) in the proof of Theorem 1, we can see that this upper bound for correct detection between m mixed quantum states can be achieved if, and only if there exists a POVM $\{\hat{\Pi}_j : 1 \leq j \leq m\}$ such that

$$\langle \phi_k^{(ij)} | \hat{\Pi}_j | \phi_k^{(ij)} \rangle = 1 \quad (51)$$

and

$$\langle \varphi_l^{(ij)} | \hat{\Pi}_j | \varphi_l^{(ij)} \rangle = 0 \quad (52)$$

for any $1 \leq i < j \leq m$, and all k, l . In this subsection, we can clearly formulate this observation and give detailed proof. We first give the following lemma that is useful to our proof.

Lemma 2. Let H be a finite dimension Hilbert space. Let S be a subspace of H , and S is spanned by a finite set of some unit vectors, say $\{|\psi_j\rangle : 1 \leq j \leq k\}$. Suppose that Π is a positive semidefinite operator on H , and $\Pi \leq I$ (i.e. $I - \Pi$ is a positive semidefinite operator) but satisfies

$$\langle \psi_j | \Pi | \psi_j \rangle = 1, \quad (53)$$

for all $1 \leq j \leq k$. Then,

$$\Pi \geq P_S \quad (54)$$

where P_S is a projection operator onto S , and $\Pi \geq P_S$ means that $\Pi - P_S$ is a positive semidefinite operator.

Proof. Since Π is a positive semidefinite operator and $\Pi \leq I$, Π has a spectral decomposition of the following form:

$$\Pi = \sum_{i=1}^l a_i |a_i\rangle \langle a_i| \quad (55)$$

where $1 \geq a_i > 0$, $1 \leq i \leq l$, and $\{|a_i\rangle : 1 \leq i \leq l\}$ are orthonormal vectors. By means of Eqs. (53,55), for any $1 \leq j \leq k$, we have

$$1 = \langle \psi_j | \Pi | \psi_j \rangle \quad (56)$$

$$= \sum_{i=1}^l a_i |\langle \psi_j | a_i \rangle|^2 \quad (57)$$

$$\leq \sum_{i=1}^l |\langle \psi_j | a_i \rangle|^2 \quad (58)$$

$$\leq \langle \psi_j | \psi_j \rangle \quad (59)$$

$$= 1. \quad (60)$$

Therefore, the above inequalities (58,59) must be two equalities. Consequently, we obtain

$$\sum_{i=1}^l a_i |\langle \psi_j | a_i \rangle|^2 = \sum_{i=1}^l |\langle \psi_j | a_i \rangle|^2 \quad (61)$$

and

$$\sum_{i=1}^l |\langle \psi_j | a_i \rangle|^2 = \langle \psi_j | \psi_j \rangle \quad (62)$$

for any $1 \leq j \leq k$.

If $a_i < 1$, then from Eq. (61) it follows that $|\langle \psi_j | a_i \rangle| = 0$ for $1 \leq j \leq k$; and, by combining this result with Eq. (62) we further have

$$\sum_{i=1; a_i=1}^l |\langle \psi_j | a_i \rangle|^2 = \langle \psi_j | \psi_j \rangle \quad (63)$$

for any $1 \leq j \leq k$.

By Eq. (63) we obtain that $|\psi_j\rangle$ can be linearly represented by the vectors in $\{|a_i\rangle : a_i = 1\}$. Since S is spanned by $\{|\psi_j\rangle : 1 \leq j \leq k\}$, any vectors in S can be linearly represented by $\{|a_i\rangle : a_i = 1\}$. In other words, $\{|a_i\rangle : a_i = 1\}$ spans a subspace of H , say S_Π , satisfying $S \subseteq S_\Pi$. Therefore, $\sum_{i, a_i=1} |a_i\rangle\langle a_i|$, denoted by P_{S_Π} , is a projection operator onto S_Π . Meanwhile, $\sum_{j, a_j < 1} a_j |a_j\rangle\langle a_j|$, denoted by $\Pi_{S_\Pi^\perp}$, is a positive semidefinite operator, satisfying $\Pi_{S_\Pi^\perp} \perp P_{S_\Pi}$.

Now we rewrite the spectral decomposition of Π by regrouping its items in the following way:

$$\Pi = \sum_{i, a_i=1} |a_i\rangle\langle a_i| + \sum_{j, a_j < 1} a_j |a_j\rangle\langle a_j| \quad (64)$$

$$= P_{S_\Pi} + \Pi_{S_\Pi^\perp}. \quad (65)$$

Since $S \subseteq S_\Pi$, we have $P_S \leq P_{S_\Pi}$ and therefore $\Pi \geq P_{S_\Pi} \geq P_S$. This proof has been completed. \square

Before giving Theorem 2, we still bring in a couple of symbols. For any m mixed quantum states $\rho_1, \rho_2, \dots, \rho_m$, with the *a priori* probabilities $\eta_1, \eta_2, \dots, \eta_m$, respectively, where, as before, we assume that the all eigenvectors corresponding to all nonzero eigenvalues of $\rho_1, \rho_2, \dots, \rho_m$ span an n -dimension Hilbert space \mathcal{H} ($m \leq n$). Let $S_{ij}^{(+)}$ denote the subspace spanned by the all eigenvectors corresponding to all *positive* eigenvalues of the Hermitian operator $\Lambda_{ij} = \eta_j \rho_j - \eta_i \rho_i$, and similarly, $S_{ij}^{(-)}$ represents the subspace spanned by the all eigenvectors corresponding to all *negative* eigenvalues of Λ_{ij} . We use S_k to denote the subspace spanned by the all eigenvectors corresponding to all *positive* eigenvalues of the $k-1$ Hermitian operators $\Lambda_{1k}, \Lambda_{2k}, \dots, \Lambda_{k-1k}$, $2 \leq k \leq m$. Therefore, S_k is the subspace spanned by $\bigcup_{i=1}^{k-1} S_{ik}^{(+)}$.

With these symbols we present Theorem 2.

Theorem 2. For any m mixed quantum states $\rho_1, \rho_2, \dots, \rho_m$, with the *a priori* probabilities $\eta_1, \eta_2, \dots, \eta_m$, respectively, then there exists a POVM $\{\Pi_i : 1 \leq i \leq m\}$ such that

$$\frac{1}{m-1} \sum_{1 \leq i < j \leq m} [\eta_i + \text{Tr}(\Lambda_{ij} \Pi_j)] = \frac{1}{2} \left(1 - \frac{1}{m-1} \sum_{1 \leq i < j \leq m} \text{Tr}|\Lambda_{ij}|\right) \quad (66)$$

if and only if the following two conditions hold:

(i) For any $1 \leq i_1, i_2 < j \leq m$,

$$P_{i_1 j}^{(+)} \perp P_{i_2 j}^{(-)} \quad (67)$$

where $P_{ij}^{(+)}$ and $P_{ij}^{(-)}$ represent the projection operators onto $S_{ij}^{(+)}$ and $S_{ij}^{(-)}$, respectively.

(ii) For any $2 \leq i < j \leq m$,

$$P_i \perp P_j \quad (68)$$

where P_k denotes the projection operator onto S_k .

Proof. (If). First, in terms of the condition (ii) described by Eq. (68), we know that

$$\sum_{j=2}^m P_j \leq I, \quad (69)$$

since S_k is a subspace of \mathcal{H} , $k = 2, 3, \dots, m$, and they are pairwise orthogonal.

We still use the symbols in the proof of Theorem 1. Recall that $\Lambda_{ij} = A_{ij} - B_{ij}$ and $A_{ij} \perp B_{ij}$, where $A_{ij} = \sum_k a_k^{(ij)} |\phi_k^{(ij)}\rangle \langle \phi_k^{(ij)}|$, and $B_{ij} = \sum_l b_l^{(ij)} |\varphi_l^{(ij)}\rangle \langle \varphi_l^{(ij)}|$.

Then, we have

$$P_{ij}^{(+)} = \sum_k |\phi_k^{(ij)}\rangle \langle \phi_k^{(ij)}|, \quad (70)$$

$$P_{ij}^{(-)} = \sum_l |\varphi_l^{(ij)}\rangle \langle \varphi_l^{(ij)}|. \quad (71)$$

According to the condition (i) described by Eq. (67), for any $1 \leq i_1, i_2 < j$, we know that $|\phi_k^{(i_1 j)}\rangle$ and $|\varphi_l^{(i_2 j)}\rangle$ are orthogonal for all k and l .

We know that S_j is the subspace spanned by $\cup_k \{|\phi_k^{(ij)}\rangle : 1 \leq i < j\}$, $2 \leq j \leq m$. With Eq. (69) we can take a POVM: $\hat{\Pi}_j = P_j$ for $j = 2, 3, \dots, m$, and $\hat{\Pi}_1 = I - \sum_{j=2}^m \hat{\Pi}_j$. Then, for $1 \leq i_1 < j \leq m$, we have $\langle \phi_k^{(i_1 j)} | P_j | \phi_k^{(i_1 j)} \rangle = 1$. Meanwhile, according to condition (i) described by Eq. (67), we have $\langle \varphi_l^{(i_2 j)} | P_j | \varphi_l^{(i_2 j)} \rangle = 0$ for $1 \leq i_2 < j \leq m$.

Therefore, with this POVM, Ineq. (38) in the proof of Theorem 1 will become an equality; more exactly, we obtain that

$$\begin{aligned} & \frac{1}{m-1} \sum_{1 \leq i < j \leq m} [\eta_i + \text{Tr}(\Lambda_{ij} \hat{\Pi}_j)] \\ = & \frac{1}{m-1} \sum_{1 \leq i < j \leq m} [\eta_i + \text{Tr}(A_{ij} \hat{\Pi}_j) - \text{Tr}(B_{ij} \hat{\Pi}_j)] \end{aligned} \quad (72)$$

$$= \frac{1}{m-1} \sum_{1 \leq i < j \leq m} [\eta_i + \sum_k a_k^{(ij)} \langle \phi_k^{(ij)} | P_j | \phi_k^{(ij)} \rangle - \sum_l b_l^{(ij)} \langle \varphi_l^{(ij)} | P_j | \varphi_l^{(ij)} \rangle] \quad (73)$$

$$= \frac{1}{m-1} \sum_{1 \leq i < j \leq m} [\eta_i + \sum_k a_k^{(ij)}] \quad (74)$$

$$= \frac{1}{2} (1 + \frac{1}{m-1} \sum_{1 \leq i < j \leq n} \text{Tr} |\eta_j \rho_j - \eta_i \rho_i|), \quad (75)$$

where the last equality results from Eq. (39). As a consequence, Eq. (66) holds.

(Only if). If there exists a POVM $\{\Pi_i : 1 \leq i \leq m\}$ such that Eq. (66) holds, then there exists a POVM $\{\hat{\Pi}_j : 1 \leq j \leq m\}$ such that Eqs. (51,52) hold, that is, $\langle \phi_k^{(ij)} | \hat{\Pi}_j | \phi_k^{(ij)} \rangle = 1$ and $\langle \varphi_l^{(ij)} | \hat{\Pi}_j | \varphi_l^{(ij)} \rangle = 0$ for any $1 \leq i < j \leq m$, and all k, l . For $j = 2, 3, \dots, m$, $\cup_k \{|\phi_k^{(ij)}\rangle : 1 \leq i < j\}$ spans S_j , so, by using Lemma 2, we have

$$\hat{\Pi}_j \geq P_j, \quad (76)$$

and, therefore,

$$\sum_{j=2}^m \hat{\Pi}_j \geq \sum_{j=2}^m P_j. \quad (77)$$

Since P_j , $2 \leq j \leq m$, are some projection operators, we can conclude that $P_i \perp P_j$ for $2 \leq i < j \leq m$. Otherwise, if $P_{i_0} \perp P_{j_0}$ does *not* hold for some $2 \leq i_0 < j_0 \leq m$, then there exists state $|\Phi_{i_0}\rangle \in S_{i_0}$ such that

$$|\Phi_{i_0}\rangle = |\Phi_{j_0}\rangle + |\Phi_{j_0}^\perp\rangle, \quad (78)$$

where $0 \neq |\Phi_{j_0}\rangle \in S_{j_0}$, and $|\Phi_{j_0}^\perp\rangle$ is orthogonal to S_{j_0} . Then, with Ineq. (77) and Eq. (78) we have

$$\langle \Phi_{i_0} | \sum_{j=2}^m \hat{\Pi}_j | \Phi_{i_0} \rangle \geq \langle \Phi_{i_0} | \sum_{j=2}^m P_j | \Phi_{i_0} \rangle \quad (79)$$

$$= \sum_{j=2}^m \langle \Phi_{i_0} | P_j | \Phi_{i_0} \rangle \quad (80)$$

$$\geq \langle \Phi_{i_0} | P_{i_0} | \Phi_{i_0} \rangle + \langle \Phi_{i_0} | P_{j_0} | \Phi_{i_0} \rangle \quad (81)$$

$$= \langle \Phi_{i_0} | \Phi_{i_0} \rangle + \langle \Phi_{j_0} | \Phi_{i_0} \rangle \quad (82)$$

$$> \langle \Phi_{i_0} | \Phi_{i_0} \rangle, \quad (83)$$

which contradicts $\sum_{j=2}^m \hat{\Pi}_j \leq I$. Therefore, condition (ii) is proved.

Furthermore, we show that condition (i) holds. By combining $P_j \leq \hat{\Pi}_j$ with Eq. (52) (i.e., $\langle \varphi_l^{(ij)} | \hat{\Pi}_j | \varphi_l^{(ij)} \rangle = 0$ for any $1 \leq i < j \leq m$ and all l), we obtain that

$$\langle \varphi_l^{(ij)} | P_j | \varphi_l^{(ij)} \rangle = 0 \quad (84)$$

for any $1 \leq i < j \leq m$ and all l .

Let P_j ($2 \leq j \leq m$) have the following spectral decomposition:

$$P_j = \sum_{t=1}^{N_j} |\Phi_t^{(j)}\rangle \langle \Phi_t^{(j)}| \quad (85)$$

where $\{|\Phi_t^{(j)}\rangle : 1 \leq t \leq N_j\}$ is an orthonormal base of S_j . From Eq. (84) it follows that

$$\langle \varphi_l^{(ij)} | \Phi_t^{(j)} \rangle = 0 \quad (86)$$

for any $1 \leq i < j \leq m$ and all l and t . Since $|\phi_k^{(i'j)}\rangle \in S_j$ for any $1 \leq i' < j \leq m$, and $\{|\Phi_t^{(j)}\rangle : 1 \leq t \leq N_j\}$ an orthonormal base of S_j , we know that $|\phi_k^{(i'j)}\rangle$ can be linearly represented by $|\Phi_t^{(j)}\rangle$, $1 \leq t \leq N_j$. Therefore, by Eq. (86) we obtain

$$\langle \varphi_l^{(i_1j)} | \phi_k^{(i_2j)} \rangle = 0 \quad (87)$$

for any $1 \leq i_1, i_2 < j \leq m$ and all l and k . In other words, condition (i) described by Eq. (67) holds. So far the proof has been completed. \square

Remark 4. When $m = 2$, these two conditions described in Theorem 2 naturally hold, and $\sum_{\substack{1 \leq i < j \leq m \\ k \neq i, j}} (\eta_i \text{Tr}(\rho_i \Pi_k)) = 0$, so, in this case, the lower bound can always be achieved, and it accords with the Helstrom limit [4]. \square

Remark 5. By means of Theorem 2, we can precisely work out the minimum-error probability for ambiguously discriminating $\rho_1, \rho_2, \dots, \rho_m$, with the *a priori* probabilities $\eta_1, \eta_2, \dots, \eta_m$, respectively, if some conditions are restricted. Indeed, we will deal with this problem in the next subsection. \square

B. Analysis concerning inequality (37)

In Ineq. (37) we leave out the term $\frac{1}{m-1} \sum_{\substack{1 \leq i < j \leq m \\ k \neq i, j}} (\eta_i \text{Tr}(\rho_i \Pi_k))$. In this subsection, in terms of the conditions described in Theorem 2, we determine the value on this term. The condition (ii) in Theorem 2 says that $S_i \perp S_j$ for $2 \leq i < j \leq m$. Here we further assume that $S_1 \perp S_j$ for $2 \leq j \leq m$, as well, where S_1 denotes the support of the positive semidefinite operator $\eta_1 \rho_1$. With this assumption and conditions (i) and (ii) in Theorem 2, we can calculate $\frac{1}{m-1} \sum_{\substack{1 \leq i < j \leq m \\ k \neq i, j}} (\eta_i \text{Tr}(\rho_i \Pi_k))$, and, also obtain a upper bound on the minimum-error probability for ambiguously discriminating in the following. We describe this result by the theorem as follows.

Theorem 3. For any m mixed quantum states $\rho_1, \rho_2, \dots, \rho_m$, with the *a priori* probabilities $\eta_1, \eta_2, \dots, \eta_m$, respectively, if $S_1 \perp S_j$ for $2 \leq j \leq m$, where S_1 denotes the support of the positive semidefinite operator $\eta_1 \rho_1$, and the two conditions described in Theorem 2 hold (that is, (i) for any $1 \leq i_1, i_2 < j \leq m$, $P_{i_1 j}^{(+)} \perp P_{i_2 j}^{(-)}$ where $P_{ij}^{(+)}$ and $P_{ij}^{(-)}$ represent the projection operators onto $S_{ij}^{(+)}$ and $S_{ij}^{(-)}$, respectively; (ii) for any $2 \leq i < j \leq m$, $P_i \perp P_j$ where P_k denotes the projection operator onto S_k), then the minimum-error probability Q_A for ambiguously discriminating $\rho_1, \rho_2, \dots, \rho_m$ satisfies

$$Q_A \leq \frac{1}{2} \left(1 - \frac{1}{m-1} \sum_{1 \leq i < j \leq m} \text{Tr}|\Lambda_{ij}| \right) + \frac{1}{2(m-1)} \sum_{2 \leq i < j \leq m} (\eta_i + \eta_j - \text{Tr}|\Lambda_{ij}|). \quad (88)$$

Proof. Firstly, with the assumption that $S_1 \perp S_j$ for $2 \leq j \leq m$, we have

$$\text{Tr}(\eta_1 \rho_1 P_j) = 0 \quad (89)$$

for $2 \leq j \leq m$. Then, for $2 \leq i < j \leq m$, due to $P_i \perp P_j$ with $1 \leq i < j \leq m$, we have

$$0 \leq \text{Tr}(\eta_i \rho_i P_j) \quad (90)$$

$$= \text{Tr}[(\eta_i \rho_i - \eta_1 \rho_1) P_j] + \text{Tr}(\eta_1 \rho_1 P_j) \quad (91)$$

$$= \text{Tr}[(A_{1i} - B_{1i}) P_j] \quad (92)$$

$$= \text{Tr}(A_{1i} P_j) - \text{Tr}(B_{1i} P_j) \quad (93)$$

$$= 0 - \text{Tr}(B_{1i} P_j) \quad (94)$$

$$\leq 0 \quad (95)$$

which results in

$$\text{Tr}(\eta_i \rho_i P_j) = 0 \quad (96)$$

for $2 \leq i < j \leq m$. In terms of Eq. (89) and $\sum_{i=1}^m P_i = I$ we have

$$\sum_{j=2}^m \text{Tr}[\eta_1 \rho_1 (P_1 + P_j)] = \sum_{j=2}^m \text{Tr}(\eta_1 \rho_1 P_1) \quad (97)$$

$$= \sum_{j=2}^m \text{Tr}[\eta_1 \rho_1 (I - \sum_{i=2}^m P_i)] \quad (98)$$

$$= \sum_{j=2}^m \text{Tr}(\eta_1 \rho_1) \quad (99)$$

$$= (m-1)\eta_1. \quad (100)$$

With Eqs. (89,96) and $\text{Tr}[(\eta_i \rho_i - \eta_1 \rho_1) P_i] = \sum_k a_k^{(1i)}$ we have

$$\sum_{2 \leq i < j \leq m} \text{Tr}[(\eta_i \rho_i - \eta_1 \rho_1)(P_i + P_j)] = \sum_{2 \leq i < j \leq m} \sum_k a_k^{(1i)} \quad (101)$$

and

$$\sum_{2 \leq i < j \leq m} \text{Tr}[\eta_1 \rho_1 (P_i + P_j)] = 0. \quad (102)$$

By the above Eqs. (100,101,102) we obtain that

$$\begin{aligned} & \sum_{1 \leq i < j \leq m} \text{Tr}[\eta_i \rho_i (P_i + P_j)] \\ &= \sum_{j=2}^m \text{Tr}[\eta_1 \rho_1 (P_1 + P_j)] \\ & \quad + \sum_{2 \leq i < j \leq m} \text{Tr}[(\eta_i \rho_i - \eta_1 \rho_1)(P_i + P_j)] \\ & \quad + \sum_{2 \leq i < j \leq m} \text{Tr}[\eta_1 \rho_1 (P_i + P_j)] \end{aligned} \quad (103)$$

$$= (m-1)\eta_1 + \sum_{2 \leq i < j \leq m} \sum_k a_k^{(1i)}. \quad (104)$$

Due to

$$\text{Tr}(\Lambda_{1i}) = \sum_k a_k^{(1i)} - \sum_l b_l^{(1i)} = \eta_i - \eta_1 \quad (105)$$

and

$$\text{Tr}|\Lambda_{1i}| = \sum_k a_k^{(1i)} + \sum_l b_l^{(1i)} \quad (106)$$

we have

$$\sum_k a_k^{(1i)} = \frac{1}{2}(\eta_i - \eta_1 + \text{Tr}|\Lambda_{1i}|). \quad (107)$$

Therefore, with Eqs. (104,107) we obtain that

$$\begin{aligned} & \frac{1}{m-1} \sum_{\substack{1 \leq i < j \leq m \\ k \neq i, j}} (\eta_i \text{Tr}(\rho_i P_k)) \\ = & \frac{1}{m-1} \sum_{1 \leq i < j \leq m} \eta_i \text{Tr}(I - P_i - P_j) \end{aligned} \quad (108)$$

$$= \frac{1}{m-1} \sum_{1 \leq i < j \leq m} \eta_i - \frac{1}{m-1} \sum_{1 \leq i < j \leq m} \text{Tr}[\eta_i \rho_i (P_i + P_j)] \quad (109)$$

$$= \frac{1}{m-1} \sum_{1 \leq i < j \leq m} \eta_i - \eta_1 - \frac{1}{2(m-1)} \sum_{2 \leq i < j \leq m} (\eta_i - \eta_1 + \text{Tr}|\Lambda_{1i}|) \quad (110)$$

$$= \frac{1}{2(m-1)} \sum_{2 \leq i < j \leq m} (\eta_i + \eta_1 - \text{Tr}|\Lambda_{1i}|). \quad (111)$$

Therefore, by combining Theorem 2 with Eq. (111), we conclude that

$$\begin{aligned} & \sum_{i=1}^m \text{Tr}(\eta_i \rho_i P_i) \\ = & \frac{1}{m-1} \sum_{1 \leq i < j \leq m} [\eta_i + \text{Tr}(\Lambda_{ij} P_j)] - \frac{1}{m-1} \sum_{\substack{1 \leq i < j \leq m \\ k \neq i, j}} (\eta_i \text{Tr}(\rho_i P_k)) \end{aligned} \quad (112)$$

$$= \frac{1}{2} \left(1 + \frac{1}{m-1} \sum_{1 \leq i < j \leq m} \text{Tr}|\Lambda_{ij}| \right) - \frac{1}{2(m-1)} \sum_{2 \leq i < j \leq m} (\eta_i + \eta_1 - \text{Tr}|\Lambda_{1i}|), \quad (113)$$

which is a lower bound on the success probability for ambiguously discriminating $\{\rho_i\}$. Equivalently, $1 - \sum_{i=1}^m \text{Tr}(\eta_i \rho_i P_i)$ is a upper bound on the minimum-error probability Q_A for ambiguously discriminating $\{\rho_i\}$. Therefore, we conclude that Ineq. (88) holds, and the proof is completed. \square

From the proof of Theorem 3 we obtain some sufficient conditions on the minimum-error probability Q_A attaining the lower bound $\frac{1}{2} \left(1 - \frac{1}{m-1} \sum_{1 \leq i < j \leq m} \text{Tr}|\Lambda_{ij}| \right)$ for ambiguously discriminating $\{\rho_i\}$, which is represented by the following corollary.

Corollary 1. For any m mixed quantum states $\rho_1, \rho_2, \dots, \rho_m$, with the *a priori* probabilities $\eta_1, \eta_2, \dots, \eta_m$, respectively, if:

1. $\eta_i + \eta_1 = \text{Tr}|\Lambda_{1i}|$ for $2 \leq i \leq m-1$,
2. for any $1 \leq i_1, i_2 < j \leq m$, $P_{i_1 j}^{(+)} \perp P_{i_2 j}^{(-)}$,
3. for any $2 \leq i < j \leq m$, $P_i \perp P_j$,

4. for $2 \leq j \leq m$, $S_1 \perp S_j$, where S_1 denotes the support of the positive semidefinite operator $\eta_1 \rho_1$,

then the minimum-error probability Q_A satisfies

$$Q_A = \frac{1}{2} \left(1 - \frac{1}{m-1} \sum_{1 \leq i < j \leq m} \text{Tr} |\eta_j \rho_j - \eta_i \rho_i| \right). \quad (114)$$

□

IV. Comparison between ambiguous and unambiguous discriminations for arbitrary m mixed quantum states

First, we would like to point out that a comparison of POVMs and projective measurements in the unambiguous and ambiguous cases has recently been made in [42]. In this section, we compare the minimum-error probability of ambiguous discrimination to the inconclusive probability of unambiguous discrimination for any m mixed states under a certain condition.

For any given m mixed quantum states $\rho_1, \rho_2, \dots, \rho_m$ with the *a priori* probabilities $\eta_1, \eta_2, \dots, \eta_m$, respectively, in this section, we compare the minimum-error probability Q_A with the optimal failure probability, say Q_U , for unambiguously discriminating them, under the condition that Q_A attains the lower bound $\frac{1}{2} \left(1 - \frac{1}{m-1} \sum_{1 \leq i < j \leq m} \text{Tr} |\eta_j \rho_j - \eta_i \rho_i| \right)$. We present the main result as follows.

Theorem 4. For any m mixed quantum states $\rho_1, \rho_2, \dots, \rho_m$, with the *a priori* probabilities $\eta_1, \eta_2, \dots, \eta_m$, respectively, if the minimum-error probability Q_A equals

$$\frac{1}{2} \left(1 - \frac{1}{m-1} \sum_{1 \leq i < j \leq m} \text{Tr} |\eta_j \rho_j - \eta_i \rho_i| \right), \quad (115)$$

then

$$Q_U \geq 2Q_A \quad (116)$$

where Q_U denotes the optimal failure probability for unambiguously discriminating $\rho_1, \rho_2, \dots, \rho_m$.

Proof. Rudolph *et al.* [27] proved that a lower bound on the failure probability Q_U for unambiguously discriminating ρ_1, ρ_2 , with given prior probabilities η_1, η_2 , respectively, is

$$Q_U \geq 2\sqrt{\eta_1 \eta_2} F(\rho_1, \rho_2), \quad (117)$$

where $F(\rho, \sigma) = (\rho^{1/2} \sigma \rho^{1/2})^{1/2}$. A generalization to the case of m states has been given by Feng *et al.* [28], i.e.,

$$Q_U \geq \sqrt{\frac{m}{m-1} \sum_{i \neq j} \eta_i \eta_j F(\rho_i, \rho_j)^2}. \quad (118)$$

In the light of Cauchy-Schwarz inequality, it is easy to get

$$\sqrt{\frac{m}{m-1} \sum_{i \neq j} \eta_i \eta_j F(\rho_i, \rho_j)^2} \geq \frac{1}{m-1} \sum_{i \neq j} \sqrt{\eta_i \eta_j} F(\rho_i, \rho_j) \quad (119)$$

$$= \frac{2}{m-1} \sum_{1 \leq i < j \leq m} \sqrt{\eta_i \eta_j} F(\rho_i, \rho_j). \quad (120)$$

Thus, to show that $Q_U \geq 2Q_A$, we only need to prove that

$$\frac{2}{m-1} \sum_{1 \leq i < j \leq m} \sqrt{\eta_i \eta_j} F(\rho_i, \rho_j) \geq 2Q_A = 1 - \frac{1}{m-1} \sum_{1 \leq i < j \leq m} \text{Tr}|\Lambda_{ij}|, \quad (121)$$

where $\Lambda_{ij} = \eta_j \rho_j - \eta_i \rho_i$ as before. Equivalently, it suffices to show that

$$\frac{1}{m-1} \sum_{1 \leq i < j \leq m} \text{Tr}|\Lambda_{ij}| + \frac{2}{m-1} \sum_{1 \leq i < j \leq m} \sqrt{\eta_i \eta_j} F(\rho_i, \rho_j) \geq 1. \quad (122)$$

In terms of [40,43,44], by choosing an appropriate orthonormal base $\{|l^{(ij)}\rangle\}$ as the eigenvectors of positive semidefinite operator $\rho_j^{-1/2}(\rho_j^{1/2} \rho_i \rho_j^{1/2}) \rho_j^{-1/2}$, then

$$F(\rho_i, \rho_j) = \sum_l \sqrt{\langle l^{(ij)} | \rho_i | l^{(ij)} \rangle} \sqrt{\langle l^{(ij)} | \rho_j | l^{(ij)} \rangle}. \quad (123)$$

Denote $e_l^{(ij)} = \langle l^{(ij)} | \rho_i | l^{(ij)} \rangle$ and $f_l^{(ij)} = \langle l^{(ij)} | \rho_j | l^{(ij)} \rangle$. Due to $\sum_{1 \leq i < j \leq m} (\eta_i + \eta_j) = m-1$, we have

$$m-1-2 \sum_{1 \leq i < j \leq m} \sqrt{\eta_i \eta_j} F(\rho_i, \rho_j) = \sum_{1 \leq i < j \leq m} [\eta_i + \eta_j - 2\sqrt{\eta_i \eta_j} F(\rho_i, \rho_j)] \quad (124)$$

$$= \sum_{1 \leq i < j \leq m} \sum_l (\sqrt{\eta_i} \sqrt{e_l^{(ij)}} - \sqrt{\eta_j} \sqrt{f_l^{(ij)}})^2, \quad (125)$$

where $\text{Tr}(\rho_i) = \sum_l e_l^{(ij)} = \sum_l f_l^{(ij)} = \text{Tr}(\rho_j) = 1$ is used.

On the other hand, as before, let $\Lambda_{ij} = A_{ij} - B_{ij}$ where A_{ij} and B_{ij} are positive semidefinite operators, and $A_{ij} \perp B_{ij}$. Then, by Lemma 1 and $A_{ij} - B_{ij} = \eta_j \rho_j - \eta_i \rho_i$, we have

$$\sum_{1 \leq i < j \leq m} \text{Tr}|\Lambda_{ij}| = \sum_{1 \leq i < j \leq m} \text{Tr}|A_{ij} - B_{ij}| \quad (126)$$

$$= \sum_{1 \leq i < j \leq m} \text{Tr}(A_{ij} + B_{ij}) \quad (127)$$

$$= \sum_{1 \leq i < j \leq m} \sum_l (\langle l^{(ij)} | A_{ij} | l^{(ij)} \rangle + \langle l^{(ij)} | B_{ij} | l^{(ij)} \rangle) \quad (128)$$

$$\geq \sum_{1 \leq i < j \leq m} \sum_l |\langle l^{(ij)} | (A_{ij} - B_{ij}) | l^{(ij)} \rangle| \quad (129)$$

$$= \sum_{1 \leq i < j \leq m} \sum_l |\eta_j f_l^{(ij)} - \eta_i e_l^{(ij)}| \quad (130)$$

$$\geq \sum_{1 \leq i < j \leq m} \sum_l (\sqrt{\eta_j f_l^{(ij)}} - \sqrt{\eta_i e_l^{(ij)}})^2 \quad (131)$$

$$= m - 1 - 2 \sum_{1 \leq i < j \leq m} \sqrt{\eta_i \eta_j} F(\rho_i, \rho_j) \quad (132)$$

where the last equality follows from Eq. (125). Therefore, Ineq. (122) holds, and the proof has been completed. \square

Indeed, we can give a simpler method to show Theorem 4. We need a fact. As we know from [40], for any mixed states ρ and σ ,

$$1 - F(\rho, \sigma) \leq D(\rho, \sigma) \quad (133)$$

where $F(\rho, \sigma) = \text{Tr} \sqrt{(\rho^{1/2} \sigma \rho^{1/2})}$ and $D(\rho, \sigma) = \frac{1}{2} \text{Tr} |\rho - \sigma|$. In fact, in the proof for Ineq. (133) in [40], the traces of ρ and σ being one is not involved, and it only utilizes the positive semidefinite property of ρ and σ . Therefore, it follows the following fact, whose proof is only a repeated process step by step according to those of [40].

Fact 1. For any two positive semidefinite operators ρ and σ , we have

$$\frac{\text{Tr}(\rho) + \text{Tr}(\sigma)}{2} - F(\rho, \sigma) \leq D(\rho, \sigma), \quad (134)$$

where, as above, $F(\rho, \sigma) = \text{Tr} \sqrt{(\rho^{1/2} \sigma \rho^{1/2})}$ and $D(\rho, \sigma) = \frac{1}{2} \text{Tr} |\rho - \sigma|$.

Alternative Method for the Proof of Theorem 4: In the light of inequalities (118,119), Eq. (120) and Fact 1, we get that

$$Q_U \geq \sqrt{\frac{m}{m-1} \sum_{i \neq j} \eta_i \eta_j F(\rho_i, \rho_j)^2} \quad (135)$$

$$\geq \frac{2}{m-1} \sum_{1 \leq i < j \leq m} \sqrt{\eta_i \eta_j} F(\rho_i, \rho_j) \quad (136)$$

$$= \frac{2}{m-1} \sum_{1 \leq i < j \leq m} F(\eta_i \rho_i, \eta_j \rho_j) \quad (137)$$

$$\geq \frac{2}{m-1} \sum_{1 \leq i < j \leq m} \left(\frac{\eta_i + \eta_j}{2} - D(\eta_i \rho_i, \eta_j \rho_j) \right) \quad (138)$$

$$= 1 - \frac{1}{m-1} \sum_{1 \leq i < j \leq m} \text{Tr} |\Lambda_{ij}| \quad (139)$$

$$= 2Q_A, \quad (140)$$

where Ineq. (138) is resulted from Fact 1. \square

V. Concluding Remarks

It is a difficult problem for giving an analytical solution for ambiguously distinguishing between any m given mixed states, and only some special cases has been solved [11,12,13,14,15]. In this paper, we have derived an analytical expression of the lower bound on the minimum-error probability for ambiguously distinguishing between arbitrary m mixed states. When $m = 2$, this bound is precisely the well-known Helstrom limit [4]. Also, we have provided a lower bound on the minimum-error probability for discriminating quantum operations. Then we have further analyzed this lower bound for ambiguous discrimination of mixed states by presenting necessary and sufficient conditions related to it. Furthermore, with a restricted condition, we have worked out a upper bound on the minimum-error probability for ambiguous discrimination of mixed states. Therefore, some sufficient conditions have been presented for the minimum-error probability attaining this bound. Finally, under the condition that the minimum-error probability attains this bound, we have compared the minimum-error probability for *ambiguously* discriminating arbitrary m mixed states with the optimal failure probability for *unambiguously* discriminating the same mixed states. When $m = 2$, this relation has been proved by Herzeg and Bergou [38].

A further question worthy of consideration is comparison between unambiguous and ambiguous discriminations without any restricted conditions. Also, this lower bound we derived may be appropriately improved, since inequality (21) can be strict for some $\rho_1, \rho_2, \dots, \rho_m$. Based on the paper, another issue is to further investigate the minimum-error probability for distinguishing between quantum operations [41]. We would like to study them in the subsequent work.

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References

- [1] A. Chefles, Contemp. Phys. **41**, 401 (2000).
- [2] J.A. Bergou, U. Herzog, and M. Hillery, *Quantum State Estimation*, Lecture Notes in Physics Vol. 649 (Springer, Berlin, 2004), p. 417; A. Chefles, *ibid.* p. 467.
- [3] Y.C. Eldar and G.D. Forney, Jr., IEEE Trans. Inform. Theory **47**, 858 (2001).
- [4] C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic Press, New York, 1976).
- [5] A.S. Holevo, J. Multivariate Anal. **3**, 337 (1973).
- [6] H.P. Yuen, R.S. Kennedy, and M. Lax, IEEE Trans. Inform. Theory **IT-21**, 125 (1975).
- [7] M. Charbit, C. Bendjaballah, and C. W. Helstrom, IEEE Trans. Inform. Theory **35**, 1131 (1989).
- [8] Y.C. Eldar, A. Megretski, and G.C. Verghess, IEEE Trans. Inform. Theory **49**, 1007 (2003).
- [9] M. Osaki, M. Ban, and O. Hirota, Phys. Rev. A **54**, 1691 (1996).
- [10] M. Ban, K. Kurokawa, R. Momose, and O. Hirota, Int. J. Theor. Phys. **36**, 1269 (1997).
- [11] Y.C. Eldar and G.D. Forney, Jr., e-print arXiv: quant-ph/0211111.
- [12] S.M. Barnett, Phys. Rev. A **64**, 030303(R) (2001).

- [13] E. Andersson, S.M. Barnett, C.R. Gilson, and K. Hunter, Phys. Rev. A **65**, 052308 (2002).
- [14] C.-L. Chou and L.Y. Hsu, Phys. Rev. A **68**, 042305 (2003).
- [15] U. Herzog and J.A. Bergou, Phys. Rev. A **65**, 050305(R) (2002).
- [16] I. D. Ivanovic, Phys. Lett. **A123**, 257 (1987).
- [17] D. Dieks, Phys. Lett. **A126**, 303 (1988).
- [18] A. Peres, Phys. Lett. **A128**, 19 (1988).
- [19] G. Jaeger and A. Shimony, Phys. Lett. **A197**, 83 (1995).
- [20] A. Peres and D.R. Terno, J. Phys. A **31**, 7105 (1998).
- [21] L.M. Duan and G.C. Guo, Phys. Rev. Lett. **80**, 4999 (1998); C.W. Zhang, C.F. Li, and G.C. Guo, Phys. Lett. A **261**, 25 (1999).
- [22] A. Chefles, Phys. Lett. A **239**, 339 (1998).
- [23] A. Chefles, S.M. Barnett, Phys. Lett. A **250**, 223 (1998).
- [24] Y. Sun, J.A. Bergou, and M. Hillery, Phys. Rev. A **66**, 032315 (2002).
- [25] Y.C. Eldar, IEEE Trans. Inform. Theory **49**, 446 (2003).
- [26] D. Qiu, Phys. Lett. A **303**, 140 (2002); D. Qiu, Phys. Lett. A **309**, 189 (2003); D. Qiu, J. Phys. A: Math. Gen. **35**, 6931 (2002).
- [27] T. Rudolph, R.W. Spekkens, and P.S. Turner, Phys. Rev. A **68**, 010301(R) (2003).
- [28] Y. Feng, R.Y. Duan, and Z. Ji, Phys. Rev. A **72**, 012313 (2005).
- [29] P. Raynal, N. Lütkenhaus, and S.J. van Enk, Phys. Rev. A **68**, 022308 (2003).
- [30] U. Herzog and J.A. Bergou, Phys. Rev. A **71**, 050301(R) (2005).
- [31] X.-F. Zhou, Y.-S. Zhang, and G.C. Guo, Phys. Rev. A **75**, 052314 (2007).

- [32] U. Herzog, Phys. Rev. A **75**, 052309 (2007).
- [33] J.A. Bergou and M. Hillery, Phys. Rev. Lett. **94**, 160501 (2005).
- [34] A. Chefles and S.M. Barnett, J. Mod. Opt. **45**, 1295 (1998).
- [35] J. Fiurášek, M. Ježek, Phys. Rev. A **67**, 012321 (2003).
- [36] Y.C. Eldar, Phys. Rev. A **67**, 042309 (2003).
- [37] G.M. D'Ariano, M.F. Sacchi, and J. Kahn, Phys. Rev. A **72**, 032310 (2005).
- [38] U. Herzog and J.A. Bergou, Phys. Rev. A **70**, 022302 (2004).
- [39] R.A. Horn, C.R. Johnson, *Matrix Analysis* (Cambridge University Press, Cambridge, 1986).
- [40] M.A. Nielsen and I.L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).
- [41] M.F. Sacchi, Phys. Rev. A **71**, 062340 (2005).
- [42] M.A.P. Touzel, R.B.A. Adamson, and A.M. Steinberg, e-print arXiv: 0708.1540v2.
- [43] C.A. Fuchs, PhD thesis, Univ. of New Mexico (1995), e-print arXiv: quant-ph/9601020.
- [44] H. Barnum, C.M. Caves, C.A. Fuchs, R. Jozsa, and B. Schumacher, Phys. Rev. Lett. **76**, 2818 (1996).